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A method for measuring echoes in the multipath propagation of v.h.f. signals

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THE BRITISH BROADCASTING CORPORATION

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A METHOD FOR MEASURING ECHOES IN THE MULTIPATH PROPAGATION OF V.H.F. SIGNALS

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Summary

This report describes a method of measuring the amplitudes and delays of the echoes encountered in the multipath propagation of radio waves. The method uses a test signal transmitted on a spare line of a 405-line television signal (CCIR Standard A) and is therefore intended in the first place for use at v.h.f. The method could easily be extended to u.h.f.

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A METHOD FOR MEASURING ECHOES IN THE MULTIPATH PROPAGATION OF V.H.F. SIGNALS

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1. Introduction

The present 405-line television transmissions (Standard A) broadcast at v.h.f. will probably be withdrawn within the next few years, and the frequencies may be considered not only for television use, but as a valuable outlet for new types of broadcasting service.

One of the factors that will affect what the frequencies are used for is the incidence of multipath propagation, or 'ghosting', an effect well known to many television viewers especially in urban and hilly areas. In this report we suggest how one might measure the amplitudes and time delays of the echo signals relative to the direct signal, in order to collect statistics on multipath propagation.

The method, developed initially for Band I, could equally well be applied to other frequencies used for television.

2. Measurement limitations

We decided for convenience to use the existing Band I television transmitters to provide a test signal. Since we would want to make measurements during normal viewing hours, the test signal would have to be put on to a spare line during the field-blanking period. At the moment an Insertion Test Signal is transmitted on lines 13 and 215. Since this test signal is no longer used very often, we would arrange to replace it with our own test signal while taking measurements.

This puts two limitations on the possible range and resolution of the measurements. Firstly, the video bandwidth is 3 MHz, so the resolution between echoes is not likely to be much better than the reciprocal of the bandwidth, i.e. about 0·3 μs. Secondly, because only one line per field is used for the test signal, echoes can not be measured that are delayed from the main signal by more than one active line time (about 80 μs). In fact, if interference between the preceding line-synchronizing pulse (sync. pulse) and the test signal is to be avoided, the maximum echo delay that can be measured is about one half of an active line (i.e. about 40 μs).

3. Choice of test signal

The test signal originally proposed was a broadband pulse sent at the mid-point of a line. The received signal would then consist of a pulse half way through the line corresponding to the direct path, followed by other pulses corresponding to the echo signals. The position of the pulse is as far as possible from the adjacent sync. pulses; this allows for the longest possible echo delays before sync. pulse echoes interfere with the test pulse or before test pulse echoes are concealed by the following sync. pulse, i.e. about 40 μs, as mentioned above.

However, we felt that a single broadband pulse would not contain sufficient energy in areas of low signal strength to provide an adequate signal-to-noise ratio. So instead we decided on a test signal consisting of a series of pulses generated by an 'm' or 'pseudo-random' binary sequence.* Two complete identical sequences of 127 elements each are transmitted consecutively, one of which is shown in Fig. 1. The two possible states of the sequence are represented in the test signal by a voltage that is either at black level or at peak white level (corresponding to 0·3 V or 1·0 V respectively, for a standard video signal). Pseudo-random sequences are discussed in Section 8.1, and the choices of the number of elements and their transmission rate are discussed in Section 8.2.

With reference to Fig. 2, when the test signal arrives at the receiver together with some echoes, shown separately, one can see that as long as none of the echoes is delayed from the direct signal by more than the length of one pseudo-random sequence, than a recorded section of the signal (shown shaded) will contain a complete sequence for each echo present. When the received signal is cross-correlated with a 'clean' version of the test signal, the result will contain peaks corresponding in amplitude and delay to each of the echoes, because of the autocorrelation property of pseudo-random sequences mentioned in Section 8.1.

4. Signal processing

The object of the processing is to resolve the echoes, and hence to measure their amplitudes and delays relative

* From an idea suggested by D.E. Susans.

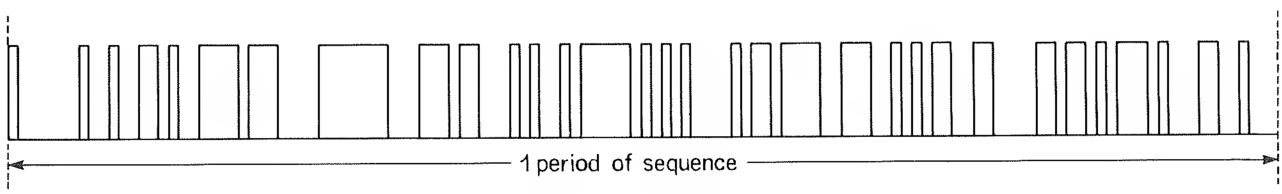

Fig. 1 - 127-element pseudo-random binary sequence

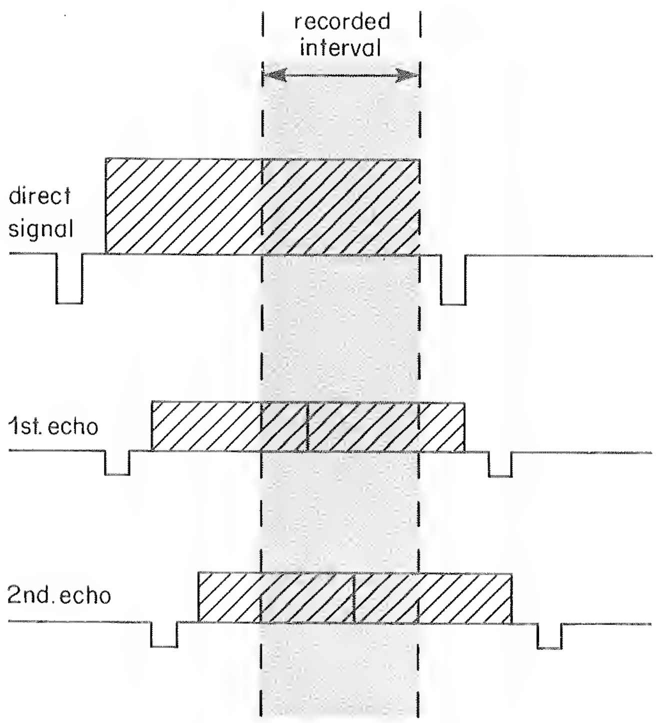

Fig. 2 - Test signal with echoes

to the main signal. The theoretical basis of the processing is discussed in Section 8.3.

The signal is recorded in digital form in a measuring vehicle after some preliminary analogue processing. The digital data is then processed on a computer back at base. We adopted this approach in an effort to keep the amount of specialized hardware to a minimum.

4.1. Signal processing in the field

A block diagram of the receiver is shown in Fig. 3. The incoming r.f. signal is first down-converted to an i.f. with a vision-carrier frequency of approximately 3·5 MHz, so that the sound-subcarrier frequency is at d.c., where it can easily be removed. The spectrum of the i.f. signal is thus of the form of Fig. 4.

The vision signal is now in a band between 0·5 MHz and 4·25 MHz. This is well within a 5·5 MHz bandwidth, so it can be sampled and converted to an 8-bit digital signal using a standard analogue-to-digital converter as used for 625-line video signals. The reason for sampling the signal at the i.f. is that it avoids having to include a vestigial sideband filter and synchronous detector in the receiver.

The sequence of 512 samples that we want (corresponding to the shaded portion of Fig. 2) is gated out and recorded. The number of samples in the sequence and the choice of sampling frequency are important and are discussed in Section 8.2.

4.2. Signal processing in the computer

The recorded data is analysed in a computer by a signal processing program. Figs. 5 to 11 were generated by this program using data that simulates the received signal.* The data is linearly interpolated for clarity. The program runs through the following stages:

4.2.1. Discrete fourier transform (d.f.t.)

The d.f.t. is described in detail in Reference 1. It is a modification of the conventional fourier series expansion to transform a finite set of sampled data and is calculated using the fast fourier transform algorithm.[2]**

Consider a periodic waveform consisting of the set of samples repeated indefinitely, such that the waveform period is an integral multiple N of the sampling interval. The fourier series expansion of the waveform will itself be periodic as a result of the sampling process. The d.f.t. of the set of sampled data is defined to be the operation that creates the set of N terms of one period of the fourier series expansion. The inverse d.f.t. performs the reverse operation; i.e. it creates one period of the original periodic waveform from its fourier series expansion.

When the received i.f. signal is sampled during the interval shown shaded in Fig. 2, the modulus of the spectrum created by the d.f.t. of the samples, if there are no echoes, is as shown in Fig. 5.

4.2.2. Demodulation

If the frequency of the carrier is such that the interval transformed by the d.f.t. contains an integral number, k_c, of cycles of the carrier, then the modulation process corresponds to shifting the spectrum of the base-band signal by $\pm k_c$ discrete spectral components. So the simplest way of demodulating the signal seems to be to shift the spectrum back again by k_c spectral components, so that the carrier frequency components become the zero frequency component. The vestigial sidebands are discarded in the process. The process is shown by the difference between Fig. 5 (before the shift) and Fig. 6 (after the shift). The inverse d.f.t. of the complex spectrum whose modulus is shown in Fig. 6 is plotted in Fig. 7, showing the 127-element pseudo-random sequence.

4.2.3. Cross-correlation

The data is correlated with the original pseudo-random sequence. The correlation is evaluated by multiplying the d.f.t. of the data by the complex conjugate of the d.f.t. of the original sequence. This produces the d.f.t. of the cross-correlation function.

The cross-correlation function itself for a signal without any echoes is shown in Fig. 8. Note that this diagram, like the other computer-generated diagrams, shows just one period of the function, which may be thought of as repeating itself indefinitely.

* Thanks are due to R.W. Lee for help in producing these diagrams.

** The Fortran program to calculate the fast fourier transform was written by the late R.E. Davies.

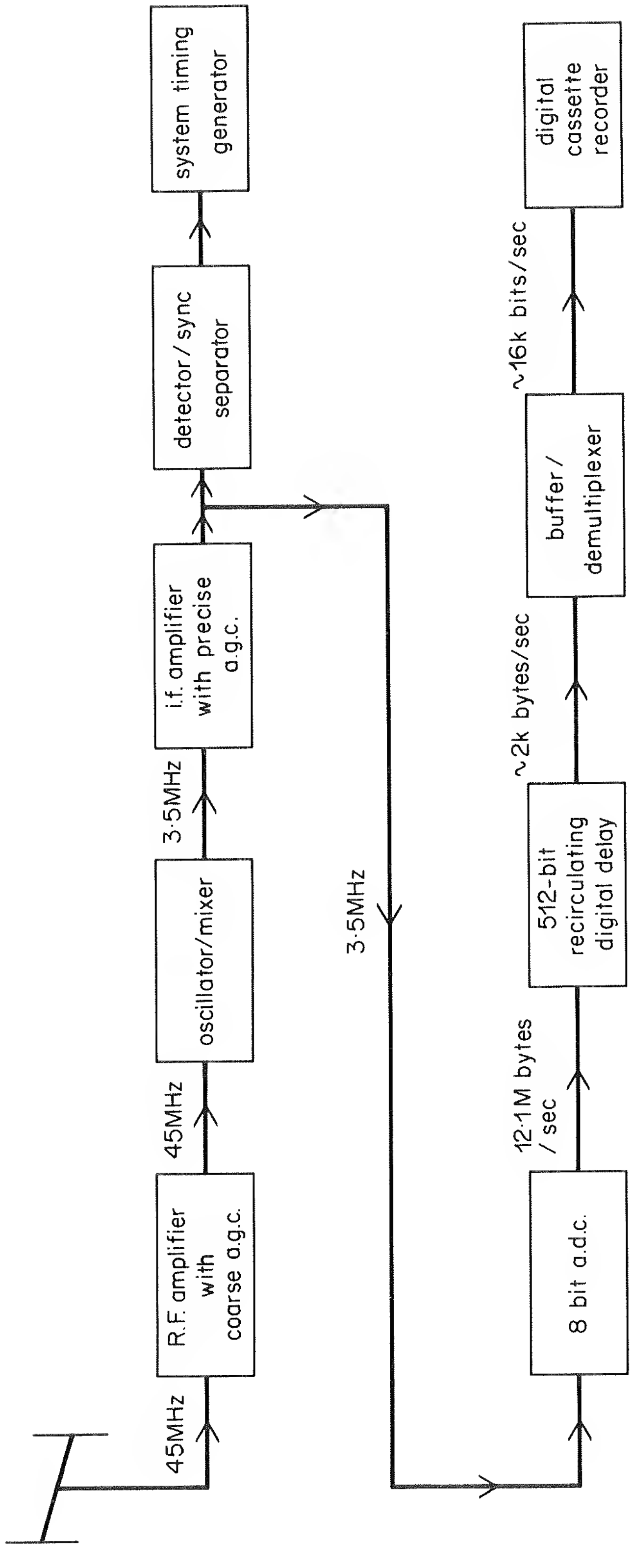

Fig. 3 - Block diagram of receiver

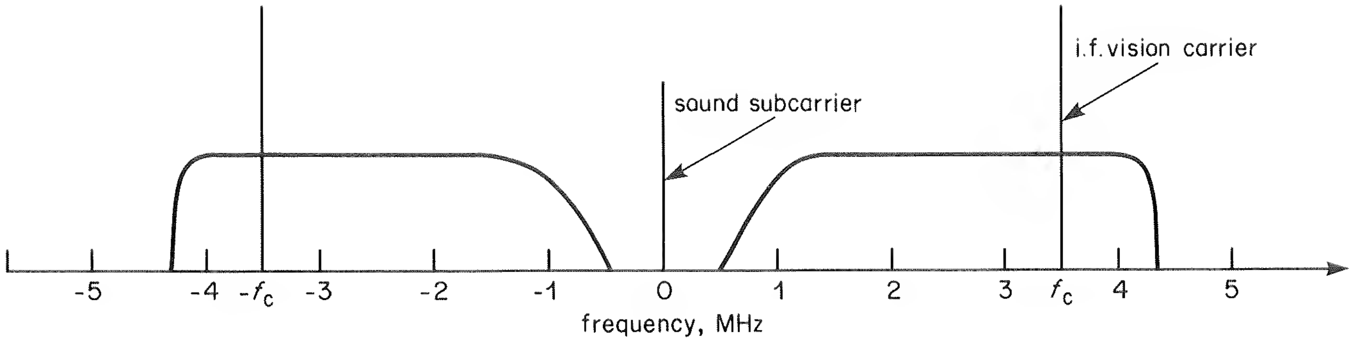

Fig. 4 - Spectrum of i.f. signal

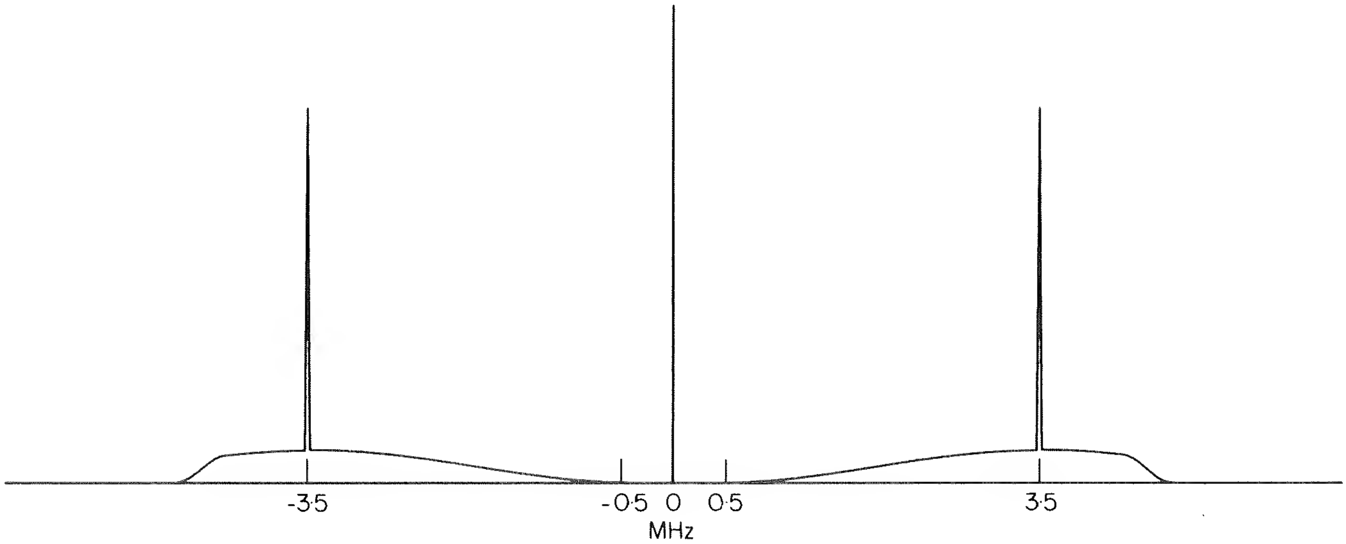

Fig. 5 - Modulus of discrete fourier transform of received signal without echoes

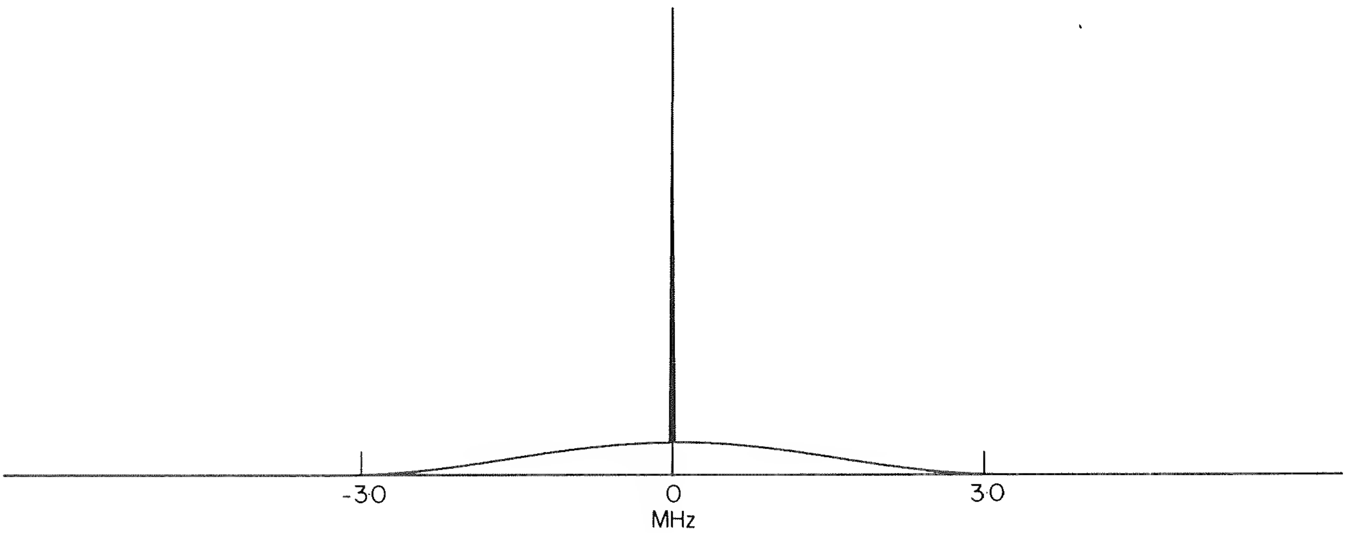

Fig. 6 - Demodulated version of Fig. 5

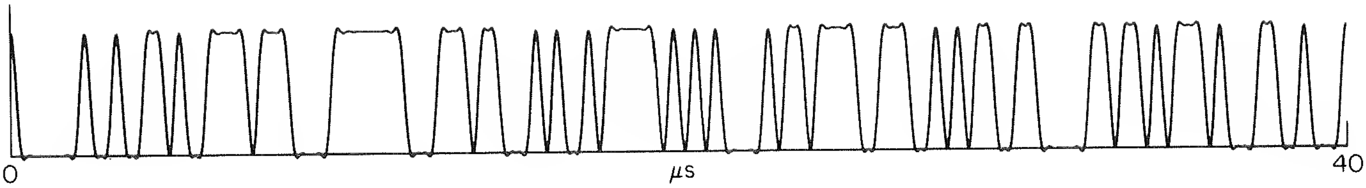

Fig. 7 - Inverse discrete fourier transform of spectrum whose modulus is shown in Fig. 6

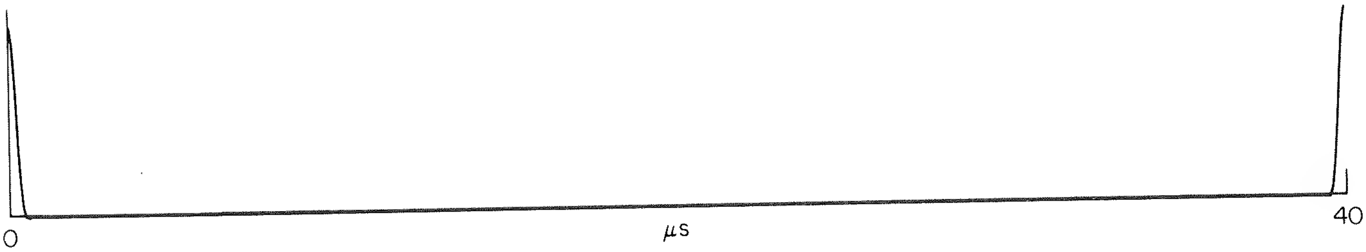

Fig. 8 - Cross-correlation of the signal of Fig. 7 with the original pseudo-random sequence

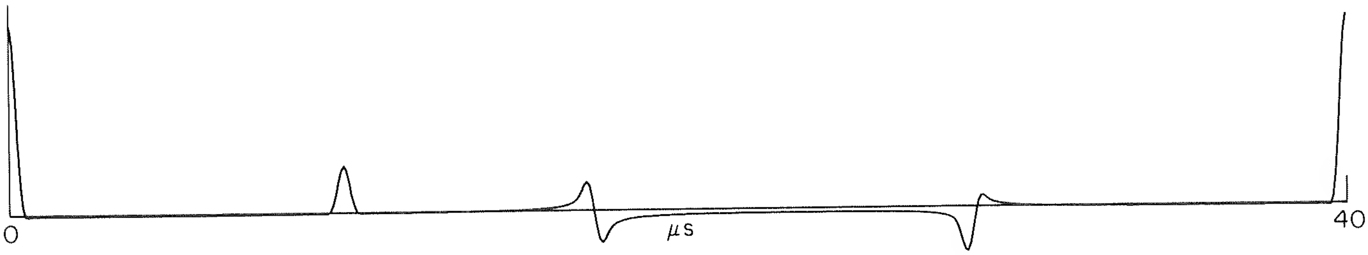

Fig. 9 - As Fig. 8 but including 3 echoes with carriers of various phases

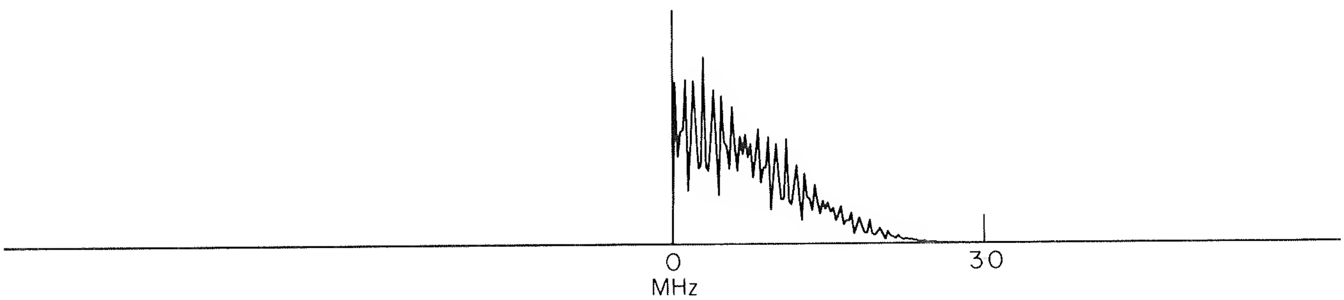

Fig. 10 - The discrete fourier transform of Fig. 9 with the negative frequencies removed

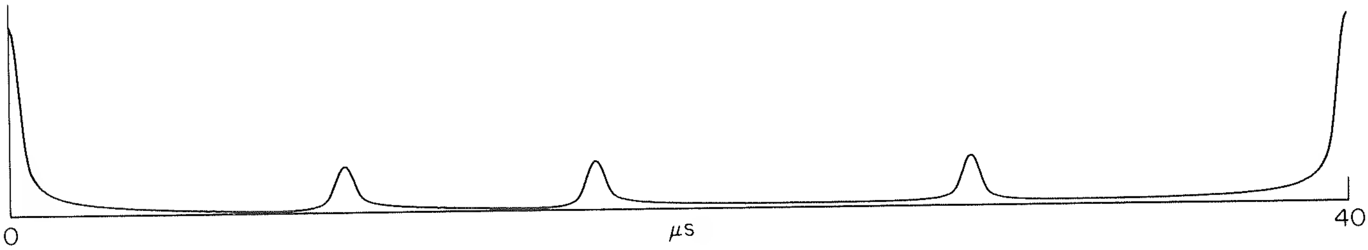

Fig. 11 - The modulus of the inverse discrete fourier transform of the spectrum whose modulus is shown in Fig. 10

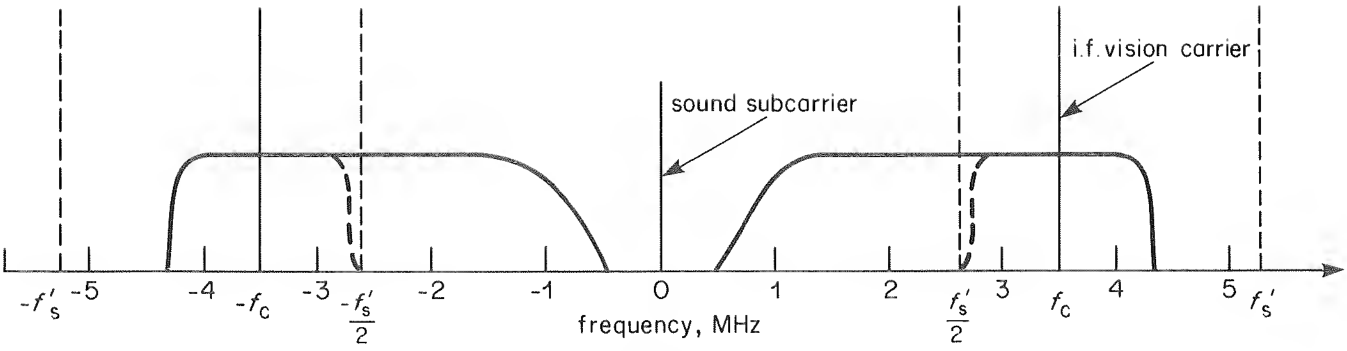

Fig. 12 - Spectrum of i.f. signal, showing reduced sampling frequency

4.2.4. The analytic signal[3,4,5]

The phases of the echo carriers will in general be random. The echoes whose carriers are in-phase (relative to the time scale of the d.f.t. interval) would be successfully demodulated by the frequency shift described above, and when the signal is transformed back into the time domain after the cross-correlation, there will be a pulse corresponding to each of these echoes. However, when those echoes whose carriers have a 90° phase shift are similarly transformed, instead of a single pulse corresponding to each echo, there will be the hilbert transform[4,5] of the pulse. The distinction is shown in Fig. 9. The carrier of the first echo was in-phase and the carrier of the second echo had a 90° phase shift. Those echoes whose carriers have an intermediate phase shift will appear as a linear combination of a pulse and its hilbert transform (e.g. the third echo of Fig. 9).

To avoid this problem, the negative frequencies are removed, leaving a spectrum of the form of Fig. 10. This is equivalent in the time domain of adding j x the hilbert transform of the signal to the signal itself. So, when the spectrum shown in Fig. 10 is transformed back into the time domain using an inverse d.f.t., a complex function is generated for each echo, whose modulus is shown in Fig. 11. This function is known as the 'analytic signal' corresponding to the signal of Fig. 9. It has the advantage that the modulus of each echo has the same shape, irrespective of the relative phases of the echo carriers.

4.2.5. Tabulating the echoes

The only technique we have tried so far for separating the echoes is to look for peaks in the modulus of the analytic signal. The highest peak is found, and the echo corresponding to this peak is subtracted from the analytic signal. The highest remaining peak is then found and the corresponding echo subtracted, and so on. With a simulated signal, this method appeared to work satisfactorily, although we are considering some techniques for improving it.

5. Longer delays

It may be necessary to be able to measure echo delays of more than half of an active line. If a complete blank line is sampled, so that the line sync. pulse is used as the test signal, then the echoes with delays of up to around 100 μs can be measured. This is equivalent to a path difference of around 30 km.

Since the sampling rate would then be reduced by a factor of around 2·5 to about 5·2 MHz, the half sampling frequency (i.e. 2·6 MHz) would lie within the passband, as shown in Fig. 12. This means that the frequency aliasing caused by the sampling would create spectrum overlap. So the signal would have to be high-pass filtered before it is sampled, with a cut-off frequency a little higher than the half sampling frequency. More severe filtering may be necessary to improve the signal-to-noise ratio, since this system no longer has the improvement in signal-to-noise ratio that was obtained by using the pseudo-random sequence. This would accordingly reduce the resolution between echoes.

If even longer echo delays are to be measured, the field sync. pulse could be used as the test signal.

6. Conclusions

The foregoing discussion indicates how one might design a system for measuring multipath parameters, using a test signal inserted on to a spare line of a System A 405-line television signal.

It appears to give reasonable resolution between echoes (around 0·3 μs, which corresponds to a path difference of around 100 m), and will measure echoes delayed by up to about 40 μs from the direct signal (i.e. a path difference of around 17 km), with the possibility of measuring delays of up to about 100 μs (i.e. 30 km path difference).

The system would require very little intervention by the operator, and all of the processing of the data may be done by a computer.

7. References

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2. Ibid, pp. 346 – 384.

3. BEDROSIAN, E. The analytic signal representation of modulated waveforms. Rand Corporation Memorandum, 1962, RM-3080-PR.

4. BRACEWELL, R.M. The Fourier transform and its application. pp. 267 – 272. McGraw-Hill Book Company, New York, 1965.

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6. GOLOMB, S.W. Shift register sequences, Part II. Holden-Day Inc., San Francisco, 1967.

7. Ibid, p.82.

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9. Ibid, pp. 69 – 70.

8. Appendix

8.1. Pseudo-random binary sequences

Pseudo-random binary sequences are periodic sequences of binary elements that have statistical properties similar to those of a truly random binary sequence; in general the more elements there are within one period of the sequence, the more closely the properties correspond. A comprehensive treatment is given in Reference 6. The most fundamental difference between pseudo-random and random sequences is that, since the pseudo-random sequences are periodic, the relevant statistical properties manifest themselves after one period of the sequence. A truly random sequence, on the other hand, would have to continue indefinitely before one could accurately measure its statistical properties.

The sequences are generated by the linear relation:

$$\pi_k = \sum_{i=1}^{m} c_i \, \pi_{k-i}$$

where the summation is modulo 2. $\{\pi_k\}$ are the elements of the sequence and $\{c_i\}$ is a set of m binary generating coefficients. Thus, by assigning arbitrary values of π_k, $k = 1,m$, then successive values of π_k, $k>m$, can be computed in turn. The relation can be implemented physically by a combination of an m-stage shift register and a modulo 2 adder connected in a feedback arrangement, for example as in Fig. 13. The i^{th} shift register stage corresponds to c_i, and a feedback path from the i^{th} shift register corresponds

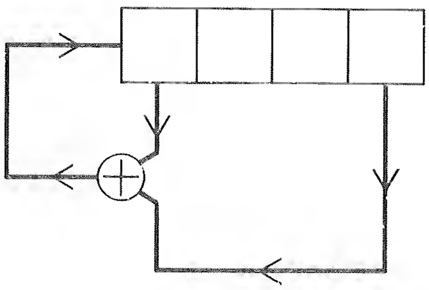

Fig. 13 - Maximal-length pseudo-random binary sequence generator of degree 4

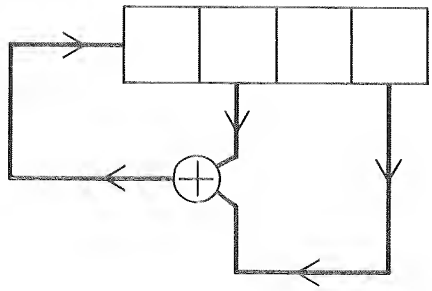

Fig. 14 - Non-maximal-length pseudo-random binary sequence generator of degree 4

to $c_i = 1$. Each different linear combination of elements, (i.e. each different feedback arrangement) will generate a different sequence. For example, the arrangement shown in Fig. 13 will generate either the sequence ....101011001000111.... or the sequence ....0000000...., according to the contents of the shift registers when the sequence is started.

If there are m stages of the shift register including in the feedback arrangement (i.e. the sequence is of degree m), then the maximum number of elements that can be generated before the sequence repeats itself is $2^m - 1$. Whether or not the sequence actually has a period of $2^m - 1$ elements depends on the choice of feedback; if it has, it is called a maximal-length sequence. For example, the arrangement of Fig. 13 produces a sequence of period 15 ($= 2^m - 1$) whereas the arrangement of Fig. 14 produces a sequence of period 6.

The property of pseudo-random sequences that is most useful to this project is that any maximal-length sequence (and also certain others) has a two-level autocorrelation function. Since the sequence elements are assigned the values 0 or 1, the autocorrelation function of such a sequence will be of the form of Fig. 15 (for a sequence of period 15), where the peak value of the function is exactly twice the lower value. So when the received signal, consisting of the sum of several sequences of various amplitudes and relative positions, is cross-correlated with an echo-free sequence, the resulting cross-correlation function will consist of peaks corresponding in amplitude and delay to each echo.

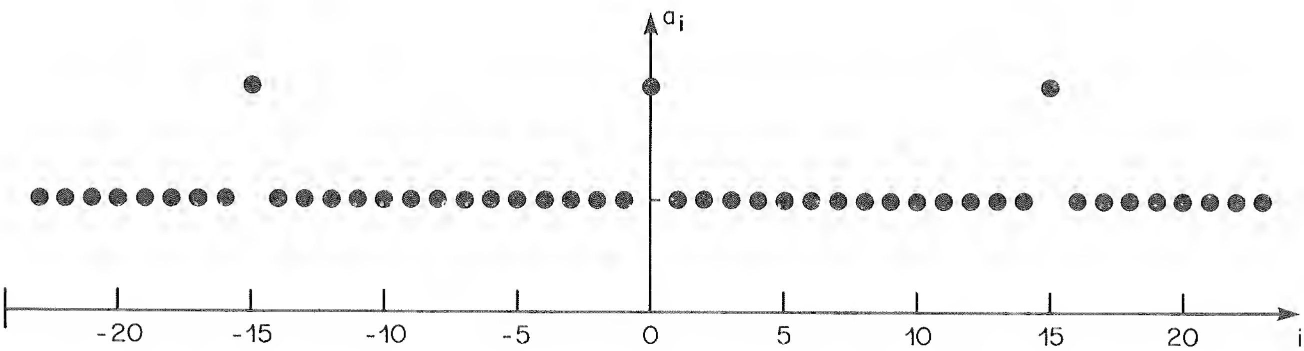

Fig. 15 - Autocorrelation function a_i of the sequence generated by the generator of Fig. 13

The particular sequence used in this project is of degree 7 and has the form:

. . . .10000001001001101001111011100001111111000111011000101001011111

01010100001011011110011100101011001100000110110101110100011001000. . . .

It is generated by the arrangement shown in Fig. 16.

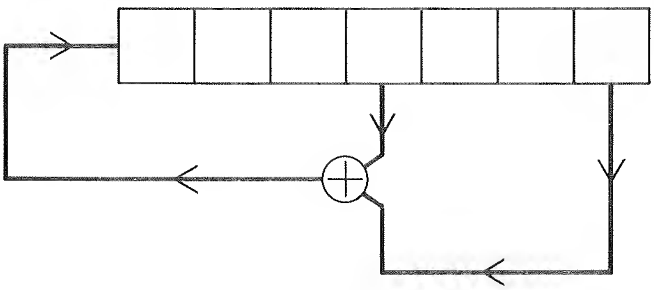

Fig. 16 - Maximal-length pseudo-random binary-sequence generator of degree 7

8.2. Choosing the signal parameters

8.2.1. Definition of parameters

f_s = sampling frequency (MHz)
f_e = element rate of pseudo-random sequence (MHz)
f_b = available bandwidth for test signal (MHz)
f_c = receiver intermediate frequency (MHz)
T = total length of one pseudo-random sequence (μs)

8.2.2. Choice of the number of elements in the pseudo-random sequence

f_b should be greater than ½f_e for all of the useful signal energy to pass through the channel; so, since f_b = 3,

$$f_e < 6 \tag{1}$$

The test signal, consisting of two pseudo-random sequences, should occupy roughly one active line of 80 μs, so

$$T \simeq 40 \tag{2}$$

Since a maximal-length pseudo-random sequence must contain $2^m - 1$ elements (m is any +ve integer), i.e.

$$T = (2^m - 1)/f_e,$$

then for the highest f_e consistent with (1) it follows from the above that m = 7, and so we have a 127-element sequence. So

$$T = 127/f_e \tag{3}$$

8.2.3. Choice of the element rate of the pseudo-random sequence

In case the test signal causes any disturbance of the pictures on the now mostly rather ancient 405 line receivers still in use, we decided to make the element rate, f_e, a multip'e of line frequency to prevent the display of the test signal jittering. So

$$f_e = k \times 0{\cdot}010125,$$

where k is an integer. Hence, using (2) and (3), k = 313.

$$\therefore \qquad f_e = 3{\cdot}169125$$

So, using (3),

$$T = 40{\cdot}074152 \tag{4}$$

8.2.4. Choice of number of samples taken and sampling rate

Since the signal processing involves the use of the fast fourier transform algorithm, it is convenient to make the number of samples a power of 2. Also we want to sample during the portion of the signal shown shaded in Fig. 2, which is T. So

$$f_s = 2^n/T \tag{5}$$

Now from Nyquist's sampling theorem, f_s must be more than twice the maximum frequency to be sampled, which here is the edge of the vestigial sideband, i.e. around 4·25 MHz; also the analogue-to-digital converter will probably not be able to sample at more than about 17 MHz. So

$$8{\cdot}5 < f_s < 17,$$

and using (4) and (5) it follows that n = 9. Hence a total number of 2^9 = 512 samples are taken, and f_s = 12·776314.

8.2.5. Choice of receiver intermediate frequency

The intermediate frequency, f_c, must be such than an exact number of complete cycles fit into the length of the sampled signal, i.e.

$$k_c/f_c = T,$$

where k_c is an integer. Also, as was mentioned earlier, it should be at about 3·5 MHz. So. using (4), it follows that k_c = 140 and f_c = 3·4935237. If f_c and f_s are maintained with an accuracy of, say, one part in 10^7, using a crystal in a temperature-controlled oven, then at worst there will be a drift of about 3×10^{-5} of a cycle for the duration of the sampling. Since the sampled signal is being quantized to 8-bit accuracy, this error will easily be masked by quantization noise.

8.3. Theoretical verification of the signal processing

8.3.1. Notation

$x(t)$ denotes a function of time with fourier transform $X(f)$. $x(t)$ and $X(f)$ are related by

$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j\,2\pi f t)\, dt,$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \exp(\ j\,2\pi f t)\, df.$$

This corresponds to system 1 in Ref. 8.

$X_T(t)$ denotes a periodic function of time with period T. If $\{X_k\}$ are the infinite fourier series expansion coefficients of $x_T(t)$, $x_T(t)$ and $\{X_k\}$ are related by

$$X_k = \int_0^T x_T(t) \exp\left(-j\frac{2\pi k t}{T}\right) dt,$$

$$x_T(t) = T \sum_{k=-\infty}^{\infty} X_k \exp\left(j\frac{2\pi k t}{T}\right).$$

If $x_T(t)$ is sampled at a sampling rate of $^N/_T$, where N is an integer, the finite set of samples $x_T\ (t + i^T/N)$, $i = 0, n - 1$, is denoted by $\langle x_i \rangle$. In this case, the fourier relations above reduce to the discrete fourier transform (d.f.t.) relations:

$$X_k = \sum_{i=0}^{N-1} x_i \exp\left(-j\frac{2\pi k i}{N}\right),$$

$$x_i = \frac{1}{N} \sum_{k=\frac{-N}{2}}^{\frac{N}{2}-1} X_k \exp\left(j\frac{2\pi k i}{N}\right).$$

$\delta_T(t)$ denotes a periodic dirac delta function, sometimes referred to as the sampling function, i.e.

$$\delta_T(t) = \sum_{i=-\infty}^{\infty} \delta(t + iT),$$

where $\delta(t)$ is the dirac delta function.[9]

δ_i is defined by

$$\delta_i = \begin{matrix} 1 \\ 0 \end{matrix},\quad i \begin{matrix} = \\ \neq \end{matrix} 0.$$

The function rect(t) is defined as

$$\text{rect}(t) = \begin{matrix} 1 \\ 0 \end{matrix},\quad |t| \lesseqgtr \tfrac{1}{2}.$$

8.3.2. Analysis of recorded signal

Let $p_T(t)$ be a periodic* maximal-length pseudo-random binary sequence of M elements π_i, $i = 0, M - 1$, with element spacing $^T/M$. The elements π_i have the value 0 or 1 depending on the value of i. The sequence can be represented by

$$p_T(t) = \sum_{i=0}^{M-1} \pi_i\, \delta_T\left(t - i\frac{T}{M}\right)$$

$$\therefore \quad P_k = \sum_{i=0}^{M-1} \pi_i \exp\left(-j\frac{2\pi k i}{M}\right).$$

From Reference 7, the autocorrelation function $a_T(t)$ is given by

$$a_T(t) = \frac{M+1}{4}\left[\delta_T(t) + \sum_{i=0}^{M-1} \delta_T\left(t - i\frac{T}{M}\right)\right].$$

$$\therefore \quad A_K = \frac{M+1}{4}\left(1 + M \sum_{n=-\infty}^{\infty} \delta_{k+nM}\right)$$

Thus

$$|P_k| = \tfrac{1}{2}\sqrt{(M+1)\left(1 + M \sum_{n=-\infty}^{\infty} \delta_{k+nM}\right)}.$$

The sequence is band-limited by a filter with a frequency response $B(f)$, typically of the form

$$B(f) = \cos\left(\frac{\pi f T}{2\,M}\right) \text{rect}\left(\frac{ft}{2M}\right),$$

and modulated on to a carrier of frequency f_c, of the form $\cos(2\pi f_c t + \phi)$, where ϕ is an arbitrary phase shift. The waveform period T is chosen such that

$$f_c T = k_c,$$

where k_c is an integer. Note that $^M/T$ is therefore the baseband signal bandwidth.

* Although the actual test signal is not repeated indefinitely, it may be thought of in this way by virtue of the finite range of the integration in the fourier series expansion relations quoted above.

The resulting waveform is passed through a vestigial sideband filter with a frequency response $V(f)$, such that

$$V(f) = 1, \; f_c - \frac{M}{T} \leqslant |f| \leqslant f_c$$

The transmitted signal will therefore consist of a waveform $w(t)$ whose fourier coefficients $\{W_k\}$ are given by

$$W_k = \left[P_{k-k_c} \, B\left(\frac{k-k_c}{T}\right) \exp(-j\phi) + P_{k+k_c} \, B\left(\frac{k+k_c}{T}\right) \exp(j\phi) \right] V\left(\frac{k}{T}\right).$$

The received signal will consist of the sum of the transmitted signal and a number of its echoes. Since the following operations are linear, we will consider just one echo $e_T(t)$ given by

$$e_T(t) = \alpha w_T(t - \tau),$$

where α, τ are the amplitude and delay respectively of the echo.

$$\therefore \qquad E_k = \alpha W_k \exp\left(-j\frac{2\pi k\tau}{T}\right).$$

This echo is then sampled at a rate $^N/_T$, where N is an integer such that

$$E_k = 0, \quad |k| \geqslant \frac{N}{2},$$

creating a set of N samples $<s_i>$.

8.3.3. Analysis of signal processing

Because of the periodic nature of data transformed by the d.f.t., all integer suffices in this section are integers modulo N.

The sampled signal has a d.f.t. spectrum $<S_k>$ given by

$$S_k = E_k, \qquad |k| < \frac{N}{2}$$

The demodulated signal $<d_i>$ is related to $<s_i>$ by the frequency shift:

$$D_k = \begin{cases} 0, & k_c \leqslant k < \dfrac{N}{2} \\ S_{k-k_c}, & 0 < k < k_c \\ \tfrac{1}{2} S_{k_c}, & k = 0 \\ 0, & -\dfrac{N}{2} \leqslant k < 0 \end{cases}$$

So substituting for S_k and simplifying, we get

$$D_k = \begin{cases} \alpha \, P_k \, B\left(\dfrac{k}{T}\right) \exp\left[j\left(\phi - \dfrac{2\pi \overline{k-k_c}\,\tau}{T}\right)\right], & 0 < k < \dfrac{N}{2} \\ \dfrac{\alpha}{2} P_o \, B\left(\dfrac{k}{T}\right) \exp\left[j\left(\phi + \dfrac{2\pi k_c \tau}{T}\right)\right], & k = 0 \\ 0 & , -\dfrac{N}{2} \leqslant k < 0 \end{cases}$$

A sampled version $<p_i>$ of $p_T(t)$ is needed to correlate $p_T(t)$ with $<d_i>$. This is created by taking values of P_k, $-^N/2 \leqslant k < {^N/2}$, as the d.f.t., $<P_k>$, of $<p_i>$. So if $<c_i>$ is the cross-correlation function of $<d_i>$ with $<p_i>$, $<C_k>$ is given by

$$C_k = D_k \cdot P_k^*$$

Substituting for D_k and P_k

$$C_k = \begin{cases} \dfrac{\alpha}{4}(M+1) \, B\left(\dfrac{k}{T}\right) \exp\left[j\left(\phi - \dfrac{2\pi \overline{k-k_c}\,\tau}{T}\right)\right], & 0 < k < \dfrac{N}{2} \\ \dfrac{\alpha}{8}(M+1)^2 \, B(0) \exp\left[j\left(\phi + \dfrac{2\pi k_c \tau}{T}\right)\right] & , k = 0 \\ 0 & , -\dfrac{N}{2} \leqslant k < 0 \end{cases}$$

If $<U_k>$ is defined as

$$U_k = \begin{cases} \pm 1, & 0 < \pm k < \dfrac{N}{2} \\ 0, & k = 0 \text{ or } \dfrac{N}{2} \end{cases},$$

then

$$C_k = \frac{\alpha}{8}(M+1)(1+U_k) \, B\left(\frac{k}{T}\right) \exp\left[j\left(\phi - \frac{2\pi \overline{k-k_c}\,\tau}{T}\right)\right]$$

$$+ \frac{\alpha}{8} M(M+1) \, \delta_k \, B(0) \exp\left[j\left(\phi + \frac{2\pi k_c \tau}{T}\right)\right].$$

$$\therefore \qquad c_i = \frac{\alpha}{8}(M+1)\exp\left[j\left(\phi + \frac{2\pi k_c \tau}{T}\right)\right]\left[\delta_i + \frac{j}{2N}\left(1-\left(-1\right)^i\right)\cot\frac{\pi i}{N}\right] * b\left(i\frac{T}{N} - \tau\right)$$

$$+\frac{\alpha}{8}M(M+1)\exp\left[j\left(\phi + \frac{2\pi k_c \tau}{T}\right)\right]B(0),$$

and

$$|c_i| = \frac{\alpha}{8}(M+1)\left|\left[\delta_i + \frac{j}{2N}\left(1-\left(-1\right)^i\right)\cot\frac{\pi i}{N}\right] * b\left(i\frac{T}{N} - \tau\right) + MB(0)\right|.$$

The symbol $*$ here denotes convolution, e.g.

$$a_i * b_i = \sum_{j=0}^{N-1} a_j \; b_{i-j}.$$

It can be seen that $\langle c_i \rangle$ is the sampled analytic signal corresponding to an echo of delay τ and waveform shape $b(t)$.

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